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WORKING PAPER SERIES N°1

« An Endogenous Approach to the Polluted River Problem »

by

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07-2025 WP

An Endogenous Approach to the Polluted River Problem

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January 16, 2025

Abstract

Polluting agents situated along a river must collectively clean it to meet a minimum water quality standard. Since cleaning operations are costly, it is crucial to allocate these expenses fairly. This paper introduces the Upstream Prorata Sharing method, which endogenously assigns responsibility to each agent based on the potential costs their pollution imposes on downstream river segments. We present three distinct axiomatic characterizations grounded in different principles. First, we explore fairness properties, which address scenarios where some agents reduce their cleanup costs. Next, we examine merging properties, analyzing the implications of two agents combining into a single entity. Finally, we propose a redistribution property, which quantifies how much an agent should contribute to cleaning downstream segments.

JEL codes: C71; D61; D62

Keywords: Polluted rivers; Cost sharing; Prorata principle; Axiomatization

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1 Introduction

1.1 Context

Consider a series of agents positioned along a linear river network, each responsible for a specific segment of the river. These agents contribute pollutants that degrade the river's ecosystem and reduce water quality. While the exact amount of pollution attributable to each agent is unknown, each agent is tasked with cleaning their segment to ensure acceptable water standards, incurring associated costs. For simplicity, we assume that water quality standards are uniform across segments, implying that agents with equally polluted segments face the same cleanup costs. In addition, we assume that the pollution is diffuse, in the sense that the pollutants emitted by a certain agent may partially transfer to downstream segments. However, the transfer rate of such diffusion is unknown. The core challenge lies in determining a fair and equitable way to allocate the total cleanup costs among all agents.

This scenario aligns with the cost-sharing model introduced by Ni and Wang [2007], which we will refer to as the polluted river problem. To address the allocation of cleanup costs, the authors proposed two methods: the Local Responsibility Sharing (LRS) and Upstream Equal Sharing (UES) methods, inspired respectively by the principles of Absolute Territorial Sovereignty and Unlimited Territorial Integrity, often invoked in international water disputes. The LRS method assigns the entire cleaning cost of a river segment to the agent occupying that segment. In contrast, the UES method distributes the cleaning cost of a segment equally among the agent in that segment and all upstream agents.

1.2 Motivating example

Consider the situation (see Figure 1) involving three agents: $\{1, 2, 3\}$, where agent 1 is the most upstream, agent 3 is the most downstream, and agent 2 is located between them. Suppose the cleaning costs are $c_1 = 0$ for agent 1, $c_2 = 4$ for agent 2, and $c_3 = 9$ for agent 3. In this situation, agent 1 produces a sufficiently low level of pollutants to stay within the water quality standard, thereby incurring no cleaning costs for its segment.

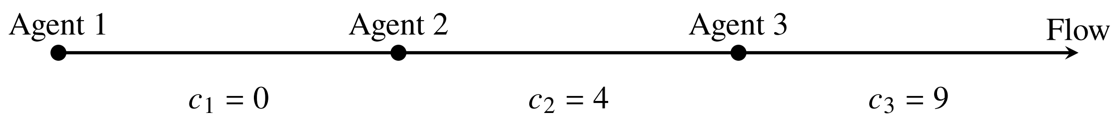

Figure 1: Example of a polluted river problem

Using the LRS method, the cost-sharing is 0 for agent 1, 4 for agent 2, and 9 for agent 3, requiring each agent to pay only for cleaning their own segment. However, a key limitation of the LRS method is that it does not consider the potential diffusion of pollutants across segments. For example, pollutants released by agent 2 may have drifted downstream, affecting the water quality in agent 3's segment. The LRS method overlooks this possibility, absolving agent 2 of any responsibility to assist agent 3 with its cleanup efforts.

In contrast, under the UES method, the cost-sharing becomes 5 for agent 1, 5 for agent 2, and 3 for agent 3. Here, the cost $c_2 = 4$ is divided equally between agents 1 and 2 because agent 1 is upstream of agent 2. Similarly, the cost $c_3 = 9$ is shared equally among all three agents. Unlike the LRS method, the UES method accounts for the diffuse nature of pollution, ensuring that upstream agents contribute to downstream cleanup efforts. However, a drawback of the UES method is that it imposes a cost of 5 on agent 1, who initially incurs no direct cleanup expenses. This could be seen as unfair, as it requires a potentially non-polluting agent to bear part of the costs for others' cleanup responsibilities.

1.3 Contribution

In this paper, we propose a new cost-sharing method that accounts for the diffuse nature of pollution while ensuring that non-polluting agents are not charged. Similar to the UES method, our method allocates the cleanup cost of each river segment among upstream agents. However, instead of dividing the costs equally, it allocates them prorata based on the cleanup costs of each agent's segments. We call it the *Upstream Prorata Sharing* (UPS) method. For example, in our motivating example, the method allocates 0 to agent 1, $4 + \frac{4}{13} \times 9$ to agent 2, and $\frac{9}{13} \times 9$ to agent 3. To illustrate, consider the cleanup cost $c_2 = 4$. While the UES method splits this equally between agents 1 and 2, the UPS allocates the cost prorata based on each agent's respective cleanup cost for their segment. However, since $c_1 = 0$, agent 2 bears the full cost of c_2. For $c_3 = 9$, the method assigns $\frac{4}{13}$ of c_3 to agent 2, $\frac{9}{13}$ to agent 3, and, again, nothing to agent 1. In this situation, agents 2 and 3 are assigned a fair portion of the cleanup costs, whereas agent 1 is exempt from any charges, as it did not incur any cleanup costs initially.

To improve interpretability and enable normative comparisons with existing methods, we present several axiomatic characterizations of the UPS method. Central to these characterizations are *fairness* axioms, which specify how the cost-sharing allocation for certain agents should adjust when downstream counterparts reduce their cleanup cost to 0. This scenario may arise if downstream agents receive external support, such as state subsidies or exogenous cleaning services, to address their river segments. Alternatively, it could reflect situations where some agents opt out of cooperative cleanup efforts and address their segments independently. We also incorporate axioms that are either well-established in the polluted river problem literature or newly introduced to reflect other allocation principles of *merging* and *cost redistribution*. One proposed axiom, based on a merging principle, asserts that if two agents combine into a single agent with aggregated cleanup costs, this should not alter the allocation of costs for their downstream counterpart. Another axiom, rooted in a cost redistribution principle, states that the share of the costs covered by an agent for each of their downstream counterparts should be redistributed among those agents in proportion to their respective cleanup costs.

1.4 Related literature

The work of Ni and Wang [2007] has inspired a rich body of literature. Several studies propose combining the LRS and UES methods through convex combinations to balance their principles [see, e.g., Sun et al., 2019, Li et al., 2023, Lowing, 2024]. These efforts aim to reconcile the competing doctrines of Absolute Territorial Sovereignty and Unlimited Territorial Integrity. Another strand of research explores the Downstream Equal Sharing (DES) method, a counterpart to the UES method, which equally divides the cost of each river segment among its downstream agents [see, e.g., Dong et al., 2012, Hou et al., 2020, 2023]. The relationship between the LRS, UES, and DES methods and cooperative game theory is comprehensively discussed in van den Brink et al. [2018]. A method closely related to ours is proposed by Gómez-Rúa [2013], which allocates the cost of each river segment among its upstream agents based on a weight system. The key distinction in our approach lies in the endogeneity of our weights. Technically, their solutions are additive, whereas ours are not, marking a fundamental methodological difference. Alcalde-Unzu et al. [2015] introduce a method that incorporates the potential transfer rate of pollution between river segments. While their motivation aligns with ours—emphasizing the cost vector's role in determining an agent's responsibility for downstream cleanup costs—their model and method center on transfer rates, a factor we exclude in this study. For a broader perspective on the polluted river problem and related models, we direct readers to the comprehensive survey by Béal et al. [2013].

1.5 Plan

The rest of the paper is organized as follows. Section 2 introduces the model of Ni and Wang [2007] as well as our cost-sharing method. Section 3 contains our axiomatic characterizations. Then, Section 4 concludes the paper, and Section 5 is an appendix containing technical details and directions for future research.

2 Preliminaries

2.1 Pollution cost-sharing problems

Consider a river divided into n segments, indexed sequentially as $i = 1, 2, \ldots, n$ from upstream to downstream. There are n agents located along a river, each positioned within one of these segments according to the specified order. We assume that each agent i generates a certain amount of pollutants, and an environmental authority enforces a pollution standard. To meet this standard, agent i must incur a cost of $c_i \geq 0$ to clean up the pollutants in segment i, ensuring the water quality complies with the environmental regulations. We aim to identify meaningful methods to allocate the total pollutant-cleaning costs $c_1 + \ldots + c_n$ among all the agents. From the perspective of responsibility, this cost allocation problem can be interpreted as determining how to distribute the total costs among the n agents, as they are accountable for the river's pollution.

Formally, a pollution cost-sharing problem is defined as a pair (N, c), where $N = \{1, \ldots, n\}$ represents the linearly ordered set of agents (and segments) and $c = (c_1, \ldots, c_n) \in \mathbb{R}_+^N$ denotes the vector of cleanup costs. Here, c_i indicates the cost for cleaning up the pollutants in segment i. For all $i, j \in N$, if $i < j$, then i is upstream of j. Conversely, if $i > j$, then i is downstream of j. Let P_N be the class of all pollution cost-sharing problems for the agent set $N = \{1, \ldots, n\}$. Let $P = \bigcup_{N \subset \mathbb{N}} P_N$ be the set of all pollution cost-sharing problem with a finite set of agents.

A cost share vector for a pollution cost-sharing problem $(N, c) \in P$ is an $|N|$-dimensional vector $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^N$ where each $x_i \geq 0$ represents the cost share allocated to agent i. A method on P is a function f that assigns a cost share vector $f(N, c) \in \mathbb{R}_+^N$ to every problem $(N, c) \in \mathcal{P}$. Two well-known methods are the Upstream Equal Sharing (UES) and the Local Responsibility Sharing (LRS). The former is defined, for each $(N, c) \in P$ and each $i \in N$ as $UES_i(N, c) = \sum_{k \geq i} c_k / |k|$; whereas the latter is simply defined as $LRS(N, c) = c$.

2.2 The UPS method

Drawing from the prorata principle illustrated in the motivating example presented in the introduction, we propose the Upstream Prorata Sharing (UPS) method. ThiS method requires an agent i to cover the cost of his downstream agents k (including i) in proportion to his own cost relative to the total cost of k's upstream agents (including k).

Definition 2.1. The method UPS is defined, for each $(N, c) \in P$, as

$$\forall i \in N, \quad UPS_i(N,c) = \begin{cases} \sum_{k \geq i} \frac{c_i}{\sum_{j \leq k} c_j} c_k & \text{if } c_i > 0, \\ 0 & \text{otherwise.} \end{cases}.$$

Equivalently, the UPS method requires an agent $i \in N$ to pay his own cost multiple times, each time in proportion to the cost of his downstream agent k (including i) relative to the total cost of k's upstream agents (including k). Thus, the local cost c_i of an agent i is delocalized to his downstream agents k (including i) acccording to the coefficient $\frac{c_k}{\sum_{j \leq k} c_j}$ which depends on cost c_k and the total cost from the source to agent k. Consequently, the UPS method can be interpreted as an endogenous approach to weighted rules proposed in Gómez-Rúa [2013] and represents a new compromise between the doctrines of Absolute Territorial Sovereignty and Unlimited Territorial Integrity.

Moreover, we observe that

$$\forall i \in N \quad \frac{\partial}{\partial c_i} UPS_i(N,c) = \frac{c_i\left(c_i + 2\sum_{j<i} c_j\right)}{\left(\sum_{j \leq i} c_j\right)^2} + \sum_{k>i} \frac{c_k \sum_{j \leq k, j \neq i} c_j}{\left(\sum_{j \leq k} c_j\right)^2},$$

$$\forall \ell < i, \quad \frac{\partial}{\partial c_\ell} UPS_i(N,c) = -\sum_{k \geq i} \frac{c_i c_k}{\left(\sum_{j \leq k} c_j\right)^2},$$

$$\text{and} \quad \forall \ell > i, \quad \frac{\partial}{\partial c_\ell} UPS_i(N,c) = \frac{c_i \sum_{j<\ell} c_j}{\left(\sum_{j \leq \ell} c_j\right)^2} - \sum_{k>\ell} \frac{c_i c_k}{\left(\sum_{j \leq k} c_j\right)^2}.$$

The cost share of any agent $i \in N$ under the UPS method is non-decreasing with respect to his own cleaning cost c_i. It is non-increasing with respect to the cleaning cost of any upstream agent c_ℓ, $l < i$. Furthermore, observe that for all $\ell, m \in N$ such that $\ell, m < i$, it holds that

$$\frac{\partial}{\partial c_\ell} UPS_i(N,c) = \frac{\partial}{\partial c_m} UPS_i(N,c).$$

Finally, the cost share of agent $i \in N$ is non-decreasing with respect to the cleaning cost of any downstream agent c_ℓ, $l > i$ if and only if

$$\frac{\sum_{j<\ell} c_j}{\left(\sum_{j\leq \ell} c_j\right)^2} \geq \sum_{k>\ell} \frac{c_k}{\left(\sum_{j\leq k} c_j\right)^2}.$$

Thus, for any $i \in N \setminus \{n\}$, $\frac{\partial}{\partial c_n} UPS_i(N, c) \geq 0$ which ensures that the cost share of any agent $i \in N \setminus \{n\}$ under the UPS method is non-decreasing with respect to the cleaning cost of the last agent n along the river.

3 Characterizations of the UPS method

In this section, we provide several axiomatic characterizations based on fairness, merging and cost redistribution principles. The section is divided into three subsection, each dedicated to a given principle.

3.1 Two characterizations based on a (weak) fairness principle

We first introduce the following axioms.

Efficiency (E). For each $(N, c) \in P$, $\sum_{i\in N} f_i(N, c) = \sum_{i\in N} c_i$.

Weak No Blind Cost (WNBC). For each $(N, c) \in P$ and $i \in N$ such that $c_j = 0$ for each $j \geq i$, $f_i(N, c) = 0$.

Proportional Fairness (PF). For each $(N, c) \in P$ and $k < n$,

$$\forall i, j \leq k+1, \quad c_j \times \left(f_i(N, c) - f_i(N, \bar{c}^k)\right) = c_i \times \left(f_j(N, c) - f_j(N, \bar{c}^k)\right),$$

where $\bar{c}^k = (c_1, c_2, \ldots, c_k, 0, \ldots, 0)$.

Efficiency (E) is a classical axiom in cooperative game theory and cost-sharing literature. It simply states that the total cost of cleaning the polluted river must be fully covered by the set of agents. Weak No Blind Cost (WNBC) is a weaker version of the No Blind Cost (NBC) axiom introduced by Ni and Wang [2007]. It asserts that an agent should not be allocated any cost if the cleaning costs of any of his downstream agent (including himself) are zero. Proportional Fairness (PF) ensures that cutting the river at a certain point, i.e. removing downstream agents from that point, leads to cost changes for the remaining agents that are proportional to their respective cleaning costs. Combining these axioms leads to a characterization of the UPS method on any class of polluter river problems with a finite set of agents.

Theorem 3.1. The UPS method is the only method on P satisfying Efficiency (E), Weak No Blind Cost (WNBC), and Proportional Fairness (PF).

Proof. We omit proving the existence part of the proof as it is direct. To show uniqueness, we proceed by induction. Consider any f on P satisfying Efficiency (E), Proportional Fairness (PF), and Weak No Blind Cost (WNBC). Consider any $(N, c) \in P$ and let us show, by induction, that $f(N, c)$ is uniquely determined. Consider $(N, \bar{c}^1) = (c_1, 0, \ldots, 0)$. By (E) and (WNBC), it holds that $f(N, \bar{c}^1) = \bar{c}^1$. Assume that there exists a $k < n - 1$ such that $f(N, \bar{c}^k)$ is uniquely determined. Let us show that $f(N, \bar{c}^{k+1})$ is uniquely determined. By (PF), it holds that

$$\forall i, j \leq k + 1, \quad c_j \times \big(f_i(N, \bar{c}^{k+1}) - f_i(N, \bar{c}^k)\big) = c_i \times \big(f_j(N, \bar{c}^{k+1}) - f_j(N, \bar{c}^k)\big).$$

With (E), we can generate $k + 1$ linearly independent equations containing the $k + 1$ unknowns $f_1(N, \bar{c}k + 1), \ldots, f_{k+1}(N, \bar{c}^{k+1})$. These equations form a solvable system of equations with a unique solution. Since $f_i(N, \bar{c}^{k+1}) = 0$ for each $i > k + 1$, we obtain that $f(N, \bar{c}^{k+1})$ is uniquely determined. This concludes the induction and the proof of the theorem. □

A second characterization, also based on a (weak) fairness principle, relies on a desirable property that clean parts of the river (if they exist) should not influence the cost allocation among agents located in the polluted segments.

Bridge (B). For each $(N, c) \in P$ and $k \leq n$, if $c_k = 0$, then

$$\forall i \neq k, \quad f_i(N \setminus \{k\}, c_{N\setminus\{k\}}) = f_i(N, c).$$

Weak Proportional Fairness (WPF). For each $(N, c) \in P$,

$$\forall i, j \leq n, \quad c_j \times \big(f_i(N, c) - f_i(N, \bar{c}^{n-1})\big) = c_i \times \big(f_j(N, c) - f_j(N, \bar{c}^{n-1})\big),$$

where $\bar{c}^{n-1} = (c_1, c_2, \ldots, c_{n-1}, 0)$.

Bridge (B) states that any agent located in a non-polluted river segment can be excluded from the pollution cost-sharing problem without impacting the cost allocation for the other agents along the river. Weak Proportional Fairness (WPF) corresponds to Proportional Fairness (PF) restricted to the case where $k = n - 1$, and therefore has a similar interpretation. Combining these two axioms with Efficiency (E) leads to a characterization the the UPS method.

Proposition 3.2. If a method satisfies Efficiency (E) and Bridge (B), then it also satisfies Weak No Blind Cost (WNBC).

Proof. Let $(N, c) \in P$ and $i \in N$ such that $c_j = 0$ for each $j \geq i$. Let us show that $f_i(N, c) = 0$. By (B), $f_k(N, c) = f_k(N \setminus \{i, \ldots, n\}, c_{N\setminus\{i,\ldots,n\}})$ for each $k < i$. Consequently,

$$\begin{aligned}\sum_{k<i} f_k(N, c) &= \sum_{k<i} f_k(N \setminus \{i, \ldots, n\}, c_{N\setminus\{i,\ldots,n\}}) \\ &= ||c_{N\setminus\{i,\ldots,n\}}|| \\ &= ||c||.\end{aligned}$$

The last equality follows from the fact that $||c|| = ||c_{N\setminus\{i,\ldots,n\}}||$, since $c_i, \ldots, c_n = 0$. Therefore, by (E), we obtain the desired result $f_i(N, c) = 0$. □

Theorem 3.3. The UPS method is the only method on P satisfying Efficiency (E), Bridge (B), and Weak Proportional Fairness (WPF).

Proof. We omit proving the existence part of the proof as it is direct. To show uniqueness, consider any f on P satisfying Efficiency (E), Bridge (B), and Weak Proportional Fairness (PF). By Proposition 3.2, f satisfies Weak No Blind Cost (WNBC). Consider any $(N, c) \in P$ and let us show that $f(N, c)$ is uniquely determined. We proceed by induction on $|N|$. If $|N| = 1$, then uniqueness is direct by (E). Assume that there is an integer $m \geq 2$, such that $f(N, c)$ is uniquely determined whenever $|N| \leq m - 1$. Let us show that $f(N, c)$ is uniquely determined if $|N| = m$. By (WPF),

$$\forall i \leq m, \quad f_i(N, c) = \frac{c_i}{c_m} \times \left(f_m(N, c) - f_m(N, \bar{c}^{m-1})\right) + f_i(N, \bar{c}^{m-1}). \tag{1}$$

By (WNBC), we have $f_m(N, \bar{c}^{m-1}) = 0$. Additionally, by (B), we have $f_i(N, \bar{c}^{m-1}) = f_i(N \setminus \{m\}, c_{N\setminus\{m\}})$. Since $|N\setminus\{m\}| = m-1$, the induction hypothesis implies that $f_i(N, \bar{c}^{m-1})$ is determined. Therefore, one can generate $m - 1$ linearly independent equations of type (1), one for each $i < m$. Adding (E), one obtains a solvable system of m equations with m unknowns. Therefore, $f(N, c)$ is uniquely determined. This concludes the induction proof. □

3.2 A characterization based on a merging fairness principle

A third characterization, based on a merging principle, is founded on the desirable property that any merging of agents should not affect the cost allocation among downstream agents.

Invariance to Upstream Merging (IUM). For each $(N, c) \in P$ and each $(N', c') \in P_{N'}$ such that $N' = (N \setminus \{ij\}) \cup \{k\}$, where $j = i + 1$, and $c'_l = c_l$ for every $l \in N \setminus \{ij\}$ and $c'_k = c_i + c_j$,

$$\forall h > i, j, f_h(N', c') = f_h(N, c).$$

Merging Proportional Fairness (MPF). For each $(N, c) \in P$ and each $(N^j, c^j) \in P_{N^j}$ such that $N^j = (N \setminus \{1, \ldots, j\}) \cup \{k\}$, where $1, \ldots, j$ represents the section of the river from 1 to j, $c^j_l = c_l$ for every $l \in N \setminus \{1, \ldots, j\}$ and $c^j_k = \sum_{m=1}^{j} c_m$,

$$c_{j+1}\big(f_k(N^j, c^j) - c^j_k\big) = c^j_k f_{j+1}(N^j, c^j).$$

Invariance to Upstream Merging (IUM) states that any merging between two successive agents does not result in any change in cost allocation for their downstream agents. Merging Proportional Fairness (MPF) ensures that the net cost paid by any merging starting from the river's source, relative to the cost paid by its immediate downstream agent along the river, is proportional to their respective cleaning costs. By net cost, we mean the difference between its allocation and the cleaning cost of its own segment.

Theorem 3.4. The UPS method is the only method on P satisfying Efficiency (E), Invariance to Upstream Merging (IUM), and Merging Proportional Fairness (MPF).

Proof. We omit proving the existence part of the proof as it is direct. To show uniqueness, consider any f on P satisfying Efficiency (E), Invariance to Upstream Merging (IUM), and Merging Proportional Fairness (MPF). Consider any $(N, c) \in P$ and let us show that $f(N, c)$ is uniquely determined. For any $j \in \{1, \ldots, n-1\}$, let $(N^j, c^j) \in P_{N^j}$ such that $N^j = (N \setminus \{1, \ldots, j\}) \cup \{k\}$, where $1, \ldots, j$ represents the section of the river from 1 to j, $c^j_l = c_l$ for every $l \in N \setminus \{1, \ldots, j\}$ and $c^j_k = \sum_{m=1}^{j} c_m$. By (MPF),

$$\forall j \in \{1, \ldots, n-1\}, \quad c_{j+1}\big(f_k(N^j, c^j) - c^j_k\big) = c^j_k f_{j+1}(N^j, c^j), \tag{2}$$

and by (E),

$$\forall j \in \{1, \ldots, n-1\}, \quad f_k(N^j, c^j) + \sum_{m=j+1}^{n} f_m(N^j, c^j) = \sum_{m=1}^{n} c_m. \tag{3}$$

We begin with $j = n-1$. From (2) and (3), we obtain $f_k(N^{n-1}, c^{n-1}) = UPS_k(N^{n-1}, c^{n-1})$ and $f_n(N^{n-1}, c^{n-1}) = UPS_n(N^{n-1}, c^{n-1})$. Furthermore, by (IUM), we have $f_n(N^{n-1}, c^{n-1}) = f_n(N, c)$ and $UPS_n(N^{n-1}, c^{n-1}) = UPS_n(N, c)$. Hence, $f_n(N, c) = UPS_n(N, c)$. Next, we consider $j = n-2$. Using a similar argument and the fact that $f_n(N, c) = UPS_n(N, c)$, we deduce that $f_{n-1}(N, c) = UPS_{n-1}(N, c)$. By iterating this procedure for each $j \in \{1, \ldots, n-1\}$ from $n-1$ down to 1, we conclude that for every $k \in N$, $f_k(N, c) = UPS_k(N, c)$, thereby completing the proof. □

3.3 A characterization based on a cost redistribution principle

A fourth characterization invokes a cost redistribution principle following the removal of an agent, reallocating the portion of the cost he covered for his downstream agents among themselves.

Converse Weak No Blind Cost (CWNBC). For each $(N, c) \in P$ and $i \in N$ such that $c_j = 0$ for each $j \leq i$, $f_i(N, c) = 0$.

Prorata Cost Redistribution (PCR). For each $(N, c) \in P$ and $k < n$,

$$\forall l \geq k+1, \quad f_l(N, \underline{c}^{k+1}) - f_l(N, \underline{c}^k) = \sum_{p=l}^{n} \frac{c_l}{\sum_{m=k+1}^{p} c_m} \times \frac{c_k}{\sum_{m=k}^{p} c_m} c_p,$$

where $\underline{c}^k = (0, \dots, 0, c_k, c_{k+1}, \dots, c_n)$.

Converse Weak No Blind Cost (CWNBC) is similar to Weak No Blind Cost (WNBC) and asserts that an agent should not be allocated any cost if the cleaning costs of any of his upstream agent (including himself) are zero. Prorata Cost Redistribution (PCR) states that the portion of the cost covered by an agent for each of his downstream agents is redistributed among those agents in proportion to their respective cleaning costs.

Theorem 3.5. The UPS method is the only method on P satisfying Efficiency (E), Converse Weak No Blind Cost (CWNBC) and Prorata Cost Redistribution (PCR).

Proof. We omit proving the existence part of the proof as it is direct. To show uniqueness, we proceed by induction. Consider any f on P satisfying Efficiency (E), Converse Weak No Blind Cost (CWNBC) and Prorata Cost Redistribution (PCR). Consider any $(N, c) \in P$ and let us show that $f(N, c)$ is uniquely determined. Consider (N, c^0). By (CWNBC), it holds that $f(N, c^0) = (0, \dots, 0)$. Consider $(N, \underline{c}^n)$. By (E) and (CWNBC), it holds that $f(N, \underline{c}^n) = (0, 0, \dots, c_n)$. Assume that there exists a $k > 1$ such that $f(N, \underline{c}^{k+1})$ is uniquely determined. Let us show that $f(N, \underline{c}^k)$ is uniquely determined. By (PCR), it holds that

$$\forall l \geq k+1, \quad f_l(N, \underline{c}^{k+1}) - f_l(N, \underline{c}^k) = \sum_{p=l}^{n} \frac{c_l}{\sum_{m=k+1}^{p} c_m} \times \frac{c_k}{\sum_{m=k}^{p} c_m} c_p.$$

With (E), we can generate $n - k + 1$ linearly independent equations containing the $n - k + 1$ unknowns $f_k(N, \underline{c}^k), \dots, f_n(N, \underline{c}^k)$. These equations form a solvable system of equations with a unique solution. Since $f_i(N, \underline{c}^k) = 0$ for each $i < k$, we obtain that $f(N, \underline{c}^k)$ is uniquely determined. This concludes the induction and the proof of the theorem. □

Although Prorata Cost Redistribution (PCR) relies on a prorata cost redistribution principle, just like the UPS method, Theorem 3.5 demonstrates that the UPS method ensures consistency in cost reallocation when an agent is removed.

4 Conclusion

4.1 Summary

This paper contributes to the literature on river pollution management by introducing a new endogenous solution, the Upstream Pro-rata Sharing (UPS) method, within the framework proposed by Ni and Wang [2007]. The main advantage of the UPS method is that it avoids overburdening an upstream agent with a low cleaning cost by incorporating a prorata weight into the compensation scheme for each of his downstream agents. In this regard, it represents a new compromise between the doctrines of Absolute Territorial Sovereignty and Unlimited Territorial Integrity. We offer several axiomatic characterizations of the UPS method, grounded in the principles of (weak) proportional fairness, merging proportional fairness, and prorata cost reallocation.

4.2 Comparison with the literature

The UPS method does not satisfy the Additivity (A) axiom introduced by Ni and Wang [2007], which states that for any two cost problems (N, c) and $(N, c') \in P_N$, an additive method f should satisfy $f(N, c + c') = f(N, c) + f(N, c')$. This failure arises because the UPS method redistributes costs using endogenous weights. In contrast, most existing methods in the polluted river problem literature satisfy (A).

For example, the UES method is characterized by combining Efficiency (E), Weak No Blind Cost (WNBC), (A), and the Upstream Symmetry (US) axiom [see Ni and Wang, 2007]. (US) asserts that two agents should share equal responsibility for any common downstream counterparts.

Additionally, Ni and Wang [2007] showed that the LRS method is characterized by Efficiency (E), (A), and No Blind Cost (NBC), a stronger version of (WNBC), which ensures that agents with no costs are not allocated anything. The following table indicates which axioms, including those discussed throughout this paper, are satisfied by the UPS, UES, and LRS methods.

	(E)	(A)	(NBC)	(WNBC)	(B)	(PF)	(WPF)	(US)	(CWNBC)	(PCR)
UPS	+	−	+	+	+	+	+	−	+	+
UES	+	+	−	+	−	−	−	+	−	−
LRS	+	+	+	+	+	−	−	−	−	−

4.3 Future research on cooperative games

Interestingly, the UPS method also aligns with the Shapley value [Shapley, 1953] of an associated cost game derived from the pollution cost-sharing problem. As detailed in Section 2, under the UPS method, each member $i \in S$ is responsible for its own costs c_i weighted by $\sum_{k\geq i} \frac{c_k}{\sum_{j\leq k} c_j}$. The total responsibility of a coalition $S \subseteq N$ is therefore simply the sum of its members' responsibilities. Therefore, for any given $(N, c) \in P$, the total costs of the coalition S can be expressed as:

$$v^c(S) = \sum_{i\in S} c_i \sum_{k\geq i} \frac{c_k}{\sum_{j\leq k} c_j}.$$

Consequently, agent i's marginal contribution is $v^c(S \cup \{i\}) - v^c(S) = c_i \sum_{k\geq i} \frac{c_k}{\sum_{j\leq k} c_j}$ for all $S \subseteq N \setminus \{i\}$ (including $\emptyset$). Hence, (N, v^c) is an additive game, and the Shapley value is determined as $\phi_i(N, v^c) = v^c(\{i\}) = UPS_i(N, c)$, $i \in N$.[1] Moreover, since (N, v^c) is additive, and so (weakly concave), the Shapley value, and therefore the UPS method, is in the core of the game (N, v^c).

In conclusion, we propose a direction for future research. We have shown that the UPS method aligns with the Shapley value [Shapley, 1953] of the game (N, v^c) as defined above. Similarly to van den Brink et al. [2018], who demonstrated that the UES method is related to the Permission value (see van den Brink [2017]), it may be valuable to further investigate the connections between the UPS method and families of values from cooperative game theory.

5 Appendix

5.1 Some results around Efficiency

Since Efficiency (E) is a desirable axiom in itself, it is relevant to ask which properties ensure that a method satisfies this condition. To this end, we introduce the following two axioms satisfied by the UPS method.

[1]Ni and Wang [2007] obtained a similar result with the LRS method by defining an additive cost game.

One-Segment Polluted (OSP). For each $(N, c^{*k}) \in P$, $k \in N$, $f(N, c^{*k}) = c^{*k}$, where $c^{*k} = (0, \ldots, 0, c_k, 0, \ldots, 0)$.

Total Cost Transfer (TCT). For each $(N, c) \in P$, and $k > 1$, then

$$f_k(N, c^{*k}) - f_k(N, \bar{c}^k) = \sum_{i<k} f_i(N, \bar{c}^k) - f_i(N, \bar{c}^{k-1}),$$

where $c^{*k} = (0, \ldots, 0, c_k, 0, \ldots, 0)$ and $\bar{c}^k = (c_1, c_2, \ldots, c_k, 0, \ldots, 0)$.

One-Segment Polluted (OSP) states that when only one segment of the river is polluted, the agent located at that segment must bear the entire cost. Total Cost Transfer (TCT) asserts that when a new agent, located at some point along the river, joins the negotiation, the entirety of his cleaning cost is shared among the agents at the table.

Proposition 5.1. If a method satisfies satisfies One-Segment Polluted (OSP) and Total Cost Transfer (TCT), then it also satisfies Efficiency (E).

Proof. First, consider any f on P satisfying One-Segment Polluted (OSP) and Total Cost Transfer (TCT). We proceed by induction. Consider $(N, \bar{c}^1)$. By (OSP), it holds that $f(N, \bar{c}^1) = (c_1, 0, \ldots, 0)$. Next, consider $(N, \bar{c}^2)$. By (TCT), we have

$$f_2(N, c^{*2}) - f_2(N, \bar{c}^2) = f_1(N, \bar{c}^2) - f_1(N, \bar{c}^1),$$

which by (OSP) and the previous step gives $f_1(N, \bar{c}^2) + f_2(N, \bar{c}^2) = c_1 + c_2$. Repeating this procedure over any $k > 1$, we obtain at the final step $k = n$ that

$$f_n(N, c^{*n}) - f_n(N, c) = \sum_{i<n} f_i(N, c) - f_i(N, \bar{c}^{n-1}).$$

which by (OSP) and the previous steps gives $\sum_{i \in N} f_i(N, c) = \sum_{i \in N} c_i$ proving that f satisfies (E). □

Another desirable property stipulates that an agent located at a clean segment of the river is ensured to incur no cost.

No Blind Cost (NBC). For each $(N, c) \in P$ and $k \leq n$, if $c_k = 0$, then $f_k(N, c) = 0$.

Weak Bridge (WB). For each $(N, c) \in P$ and $k \leq n$, if $c_k = 0$, then

$$\sum_{i \in N \setminus \{k\}} f_i(N \setminus \{k\}, c_{N \setminus \{k\}}) = \sum_{i \in N \setminus \{k\}} f_i(N, c).$$

No Blind Cost (NBC) states that an agent with no cleaning costs has no cost to pay. Weak Bridge (WB) asserts that ny agent located in a non-polluted river segment can be excluded from the pollution cost-sharing problem without affecting the total cost allocation among the remaining agents. Note that both the No Blind Cost (NBC) and Weak Bridge (WB) axioms are satisfied by the UPS method.

Proposition 5.2. If a method satisfies satisfies Weak Bridge (WB) and Efficiency (E), then it also satisfies One-Segment Polluted (OSP) and Total Cost Transfer (TCT).

Proof. First, let $(N, c^{*k}) \in P$, $k \in N$. By (E), it holds that $\sum_{i \in N} f_i(N, c^{*k}) = c_k$. Moreover, (E) and (WB) imply that $f_k(N, c^{*k}) = f_k(\{k\}, (c_k)) = c_k$, and therefore f satisfies (OSP).
Second, it follows from (WB) and (E) that $\sum_{i \leq k} f_i(N, \bar{c}^k) = \sum_{i \leq k} f_i(K, c_K) = \sum_{i \leq k} c_i$ where $K = \{1, 2, \ldots, k\}$ and $c_K = (c_1, c_2, \ldots, c_k)$. Moreover, (WB) and (E), via (OSP), imply that $\sum_{i<k} f_i(N, \bar{c}^{k-1}) + f_k(N, c^{*k}) = \sum_{i<k} f_i(K-1, c_{K-1}) + c_k = \sum_{i \leq k} c_i$, thereby proving that f satisfies (TCT). □

Proposition 5.3. If a method satisfies Weak Bridge (WB) and Efficiency (E), then it also satisfies NO Blind Cost (NBC). Furthermore, if it satisfies No Blind Cost (NBC) and Efficiency (E), then it also satisfies Weak Bridge (WB).

Proof. Let $(N, c) \in P$ and $k \leq n$ such that $c_k = 0$. First, it follows from (WB) and (E) that $\sum_{i \in N \setminus \{k\}} f_i(N, c) = \sum_{i \in N \setminus \{k\}} f_i(N \setminus \{k\}, c_{N \setminus \{k\}}) = \sum_{i \in N \setminus \{k\}} c_i$. Moreover, by (E) we have $\sum_{i \in N} f_i(N, c) = \sum_{i \in N} c_i = \sum_{i \in N \setminus \{k\}} c_i$ which permits to conclude that $f_k(N, c) = 0$.
Second, it follows from (NBC) and (E) that $\sum_{i \in N \setminus \{k\}} f_i(N, c) = \sum_{i \in N} f_i(N, c) = \sum_{i \in N} c_i = \sum_{i \in N \setminus \{k\}} c_i = \sum_{i \in N \setminus \{k\}} f_i(N \setminus \{k\}, c_{N \setminus \{k\}})$, which permits to conclude that f satisfies (WB). □

5.2 Some merging properties

We propose axioms that describe the dynamics when certain agents undertake the cleaning efforts of their neighbors, thereby supporting their cleaning costs.

Invariance to Merging (IM). For each $(N, c) \in P$ and each $(N', c') \in P_{N'}$ such that $N' =$

$(N \setminus \{ij\}) \cup \{k\}$, where $j = i+1$, and $c'_l = c_l$ for every $l \in N \setminus \{ij\}$ and $c'_k = c_i + c_j$,

$$\forall h \in N \setminus \{ij\}, f_h(N', c') = f_h(N, c),$$
$$f_k(N', c') = f_i(N, c) + f_j(N, c).$$

Invariance to Upstream Merging 2 (IUM2). For each $(N, c) \in P$ and each $(N, c') \in P_{N'}$ such that $c'_l = c_l$ for every $l \in N \setminus \{ij\}$, $j = i+1$, $c'_i = c_i + c_j$, and $c'_j = 0$,

$$\forall h > j, \quad f_h(N, c') = f_h(N, c).$$

Proposition 5.4. The UPS method satisfies Invariance to Upstream Merging (IUM2), but do not satisfy Invariance to merging (IM).

Proof. Since the proof is straightforward, it is left to the reader. □

A stronger version of Merging Proportional Fairness (MPF) applies to any merging among a set of successive agents along the river, not necessarily starting at the river's source.

Strong Merging Proportional Fairness (SMPF). For each $(N, c) \in P$ and each $(N^{ij}, c^{ij}) \in P_{N^{ij}}$ such that $N^{ij} = (N \setminus \{i, \ldots, j\}) \cup \{k\}$, where $i, \ldots, j$ represents the section of the river from i to j, $c^{ij}_l = c_l$ for every $l \in N \setminus \{i, \ldots, j\}$ and $c^{ij}_k = \sum_{m=i}^{j} c_m$,

$$c_{j+1}\left(f_k(N^{ij}, c^{ij}) - c^{ij}_k\right) = c^{ij}_k f_{j+1}(N^{ij}, c^{ij}).$$

Proposition 5.5. The UPS method satisfies Strong Merging Proportional Fairness (SMPF).

Proof. Since the proof is straightforward, it is left to the reader. □

5.3 Some monotonicity properties

Monotonicity properties are often used to axiomatically characterize cooperative solutions (see, e.g., Young [1985]). We propose the following two axioms.

Cost monotonicity (CM). For each $(N, c) \in P$ and each $(N, c') \in P_{N'}$ such that $c'_i \geq c_i$ for some $i \in N$, and $c'_j = c_j$ for every $j \neq i$.

$$f_i(N, c') \geq f_i(N, c).$$

Rank monotonicity (RM). For each $(N, c) \in P$ such that $c_i = c_j$ for some $i, j \in N$ such that $i > j$.

$$f_i(N, c) \leq f_j(N, c).$$

Cost monotonicity (CM) requires that the cost share of any agent be non-decreasing with respect to his own cleaning cost. Rank Monotonicity (RM) requires that the cost shares of agents with the same cleaning costs be non-decreasing as one moves along the river.

Proposition 5.6. The UPS method satisfies both Cost monotonicity (CM) and Rank monotonicity (RM).

Proof. The UPS method satisfies Cost monotonicity (CM) since $\frac{\partial}{\partial c_i} UPS_i(N, c) \geq 0$ as established in Section 2. Moreover, it satisfies Rank monotonicity (RM) by the definition of the UPS method. □

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